

Automatic Control



Chapter Four

Mathematical Modeling of fluid and thermal systems

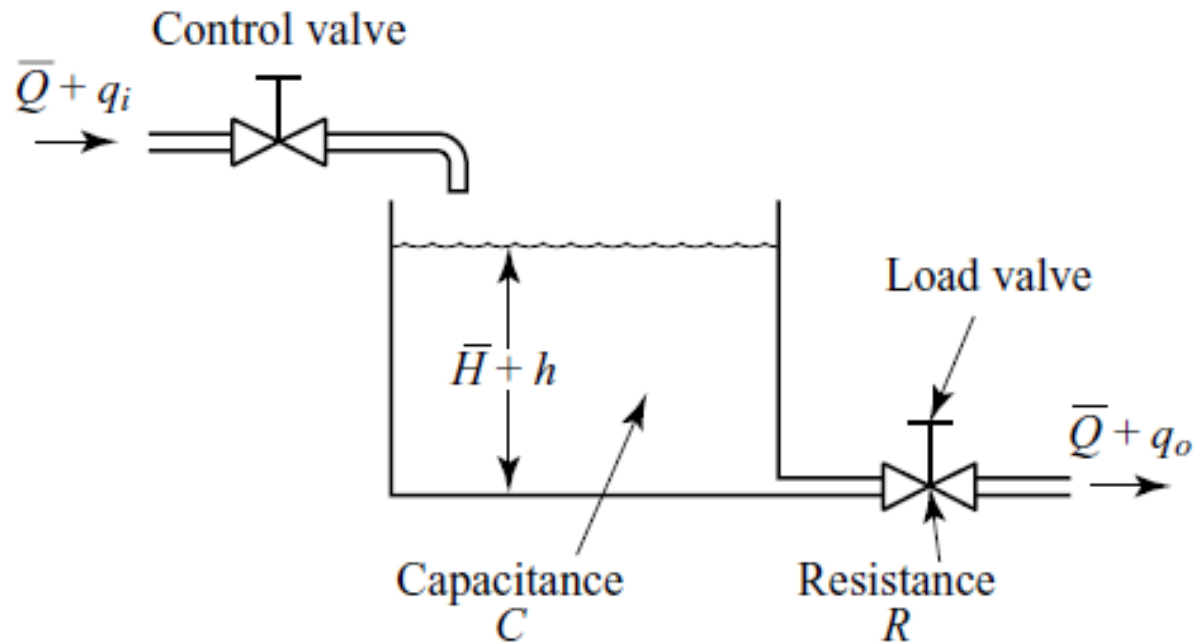
By

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4-2. LIQUID-LEVEL SYSTEMS

Resistance and Capacitance of Liquid-Level Systems

$$R = \frac{\text{change in level difference, m}}{\text{change in flow rate, m}^3/\text{sec}}$$





4-2. LIQUID-LEVEL SYSTEMS

Resistance of Liquid-Level Systems

For laminar flow $Q = KH$

where Q = steady-state liquid flow rate, m^3/sec

K = coefficient, m^2/sec

H = steady-state head, m

the resistance R_t is obtained as: $R_t = \frac{dH}{dQ} = \frac{H}{Q}$

For turbulent flow $Q = K\sqrt{H}$

where Q = steady-state liquid flow rate, m^3/sec

K = coefficient, $\text{m}^{2.5}/\text{sec}$

H = steady-state head, m

The resistance R_t for turbulent flow is obtained from

$$R_t = \frac{dH}{dQ}$$

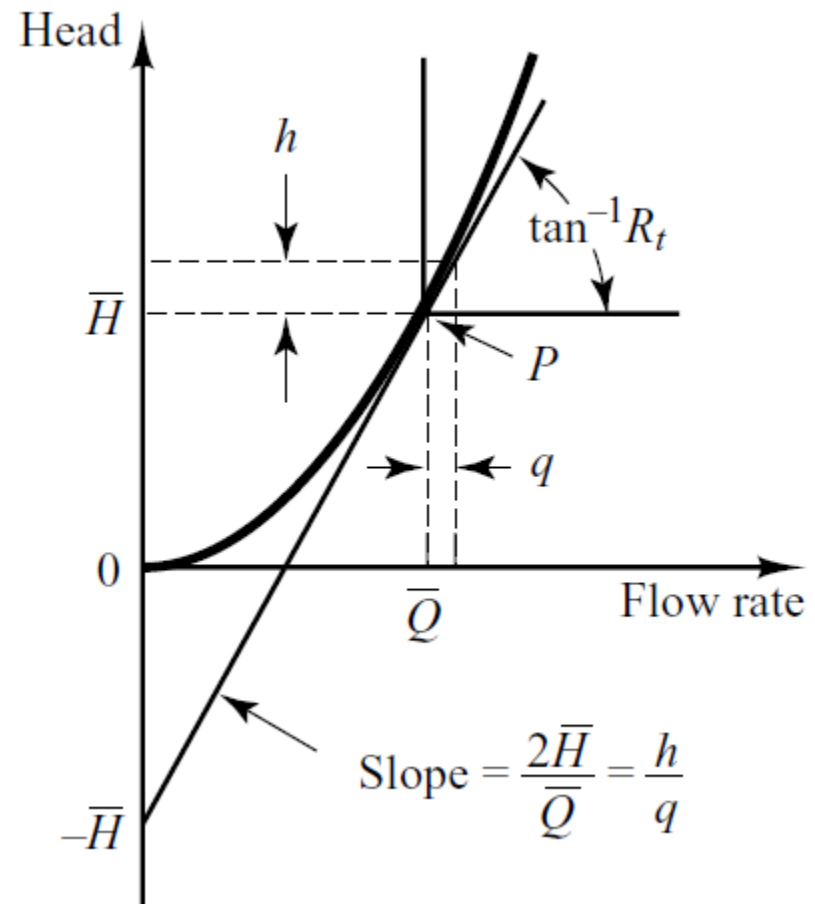
4-2. LIQUID-LEVEL SYSTEMS

The value of K is found by:

1. Conducting experiment to draw the head versus the flow rate graph.
2. Define the steady state operation point (P)
3. Draw a tangent line to H Vs Q curve from point P.
4. Find the slope of this line which represents R_t .

If a small deviation in head (h) and small deviation in flow rate (q) are assumed, the slope at point P is given as h/q which equal to

$$\frac{2\bar{H}}{\bar{Q}} = R_t$$





4-2. LIQUID-LEVEL SYSTEMS

Capacitance of Liquid-Level Systems

$$C = \frac{\text{change in liquid stored, m}^3}{\text{change in head, m}}$$

If the tank has a constant cross sectional area (A):

$$C = A$$

If the tank has a variable cross sectional area, a relation between the change in liquid stored and the change in head must be found



4-2. LIQUID-LEVEL SYSTEMS

Liquid-Level Systems.

$$\left. \begin{aligned} C \, dh &= (q_i - q_o) \, dt \\ q_o &= \frac{h}{R} \end{aligned} \right\} RC \frac{dh}{dt} + h = Rq_i \quad \Rightarrow \quad (RCs + 1)H(s) = RQ_i(s)$$

where

$$H(s) = \mathcal{L}[h] \quad \text{and} \quad Q_i(s) = \mathcal{L}[q_i]$$

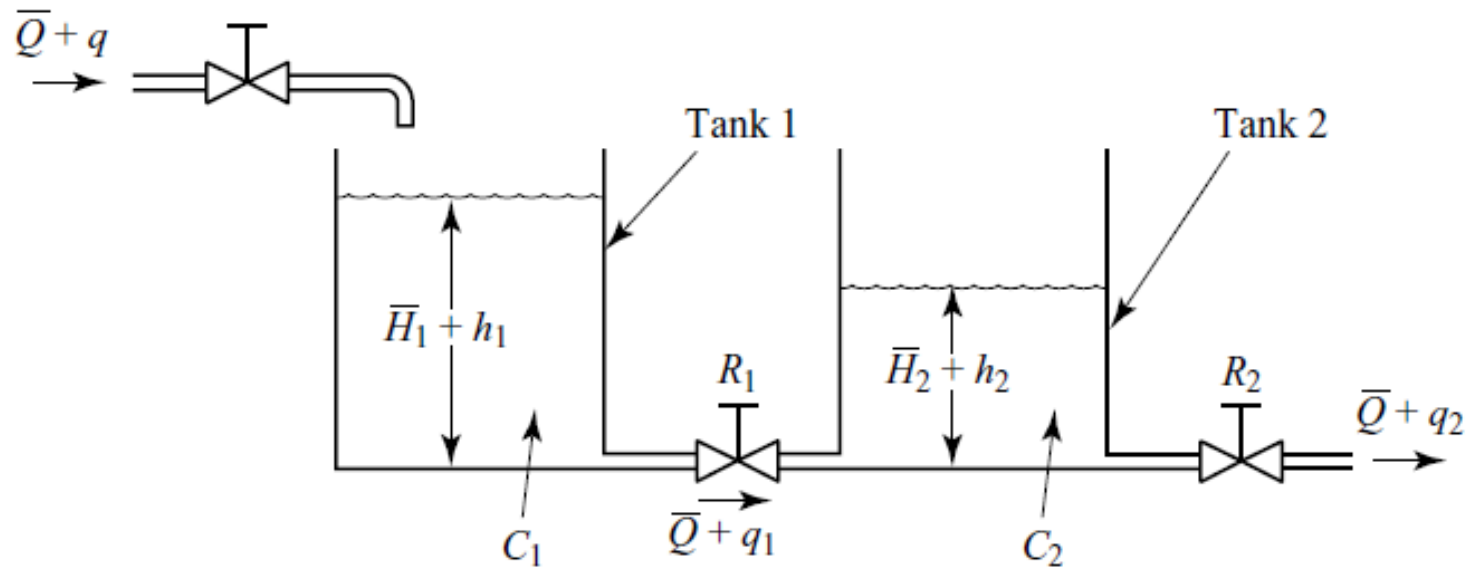
$$\frac{H(s)}{Q_i(s)} = \frac{R}{RCs + 1}$$

$$\frac{Q_o(s)}{Q_i(s)} = \frac{1}{RCs + 1}$$



4-2. LIQUID-LEVEL SYSTEMS

Liquid-Level Systems with Interaction.



\bar{Q} : Steady-state flow rate
 \bar{H}_1 : Steady-state liquid level of tank 1
 \bar{H}_2 : Steady-state liquid level of tank 2



4-2. LIQUID-LEVEL SYSTEMS

Liquid-Level Systems with Interaction.

$$\frac{h_1 - h_2}{R_1} = q_1 \quad (4-3)$$

$$C_1 \frac{dh_1}{dt} = q - q_1 \quad (4-4)$$

$$\frac{h_2}{R_2} = q_2 \quad (4-5)$$

$$C_2 \frac{dh_2}{dt} = q_1 - q_2 \quad (4-6)$$

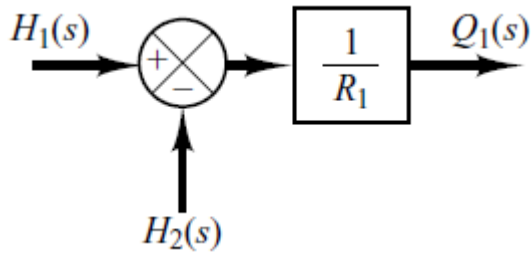
$$\frac{Q_2(s)}{Q(s)} = \frac{1}{R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_2 C_1)s + 1} \quad (4-7)$$



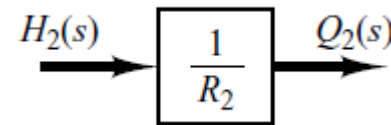
4-2. LIQUID-LEVEL SYSTEMS

Liquid-Level Systems with Interaction.

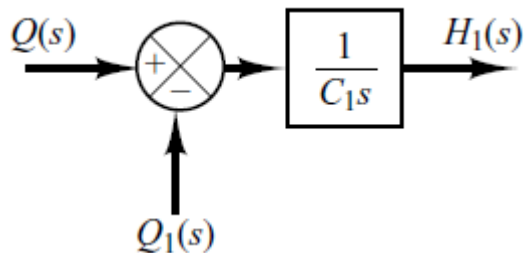
$$\frac{h_1 - h_2}{R_1} = q_1$$



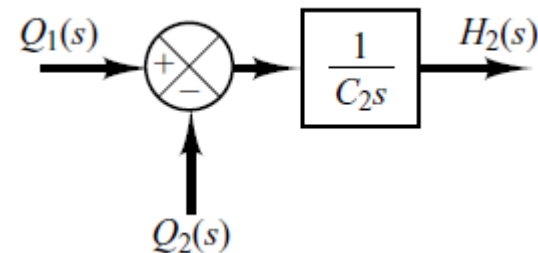
$$\frac{h_2}{R_2} = q_2$$



$$C_1 \frac{dh_1}{dt} = q - q_1$$



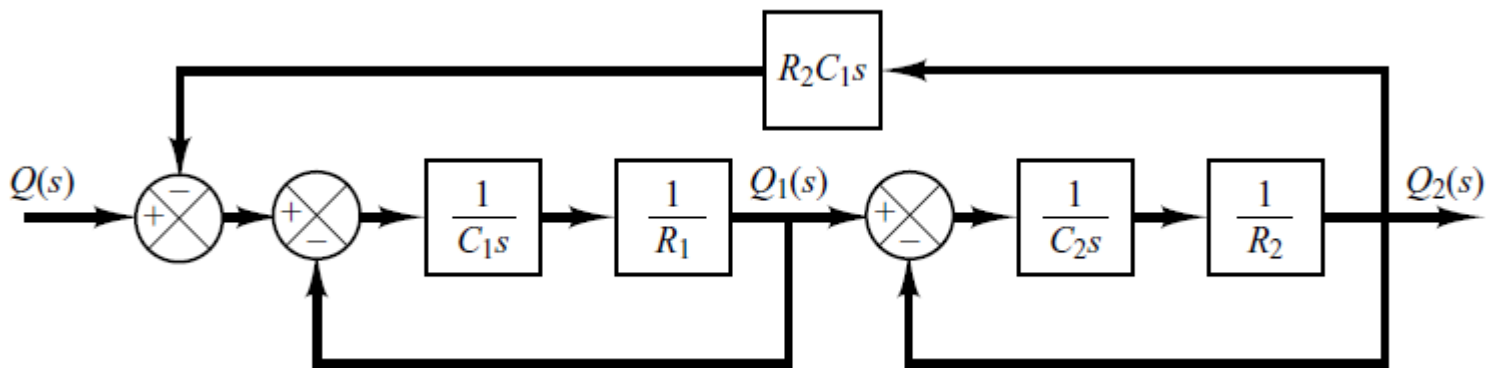
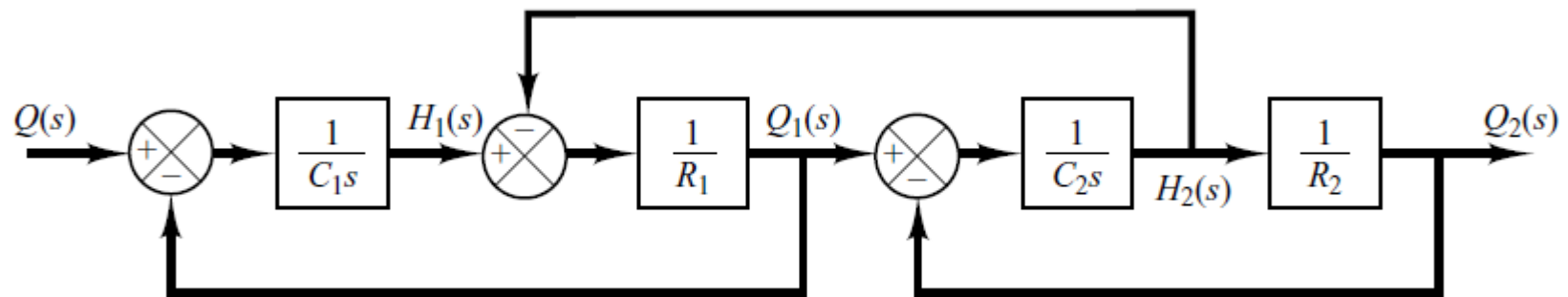
$$C_1 \frac{dh_1}{dt} = q - q_1$$





4-2. LIQUID-LEVEL SYSTEMS

Liquid-Level Systems with Interaction.

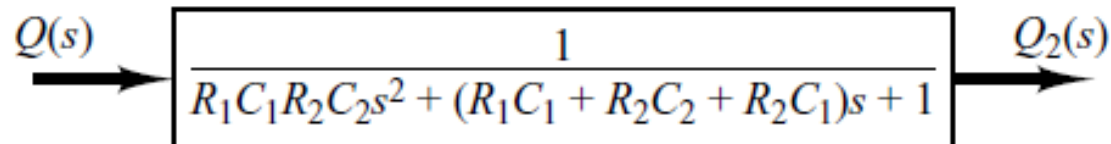
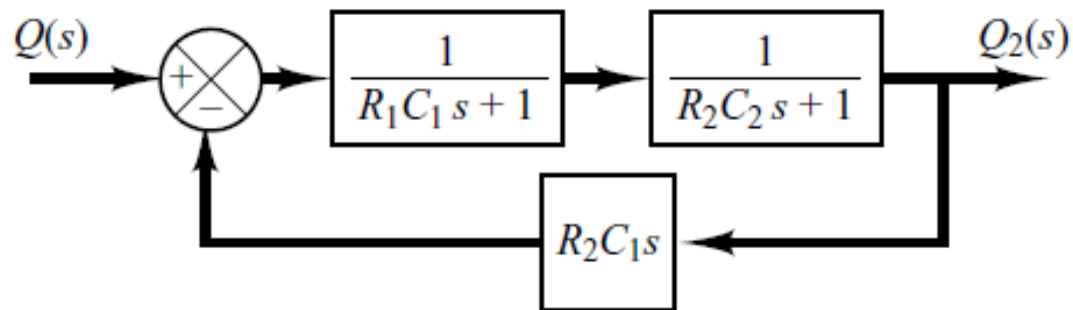


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4-2. LIQUID-LEVEL SYSTEMS

Liquid-Level Systems with Interaction.

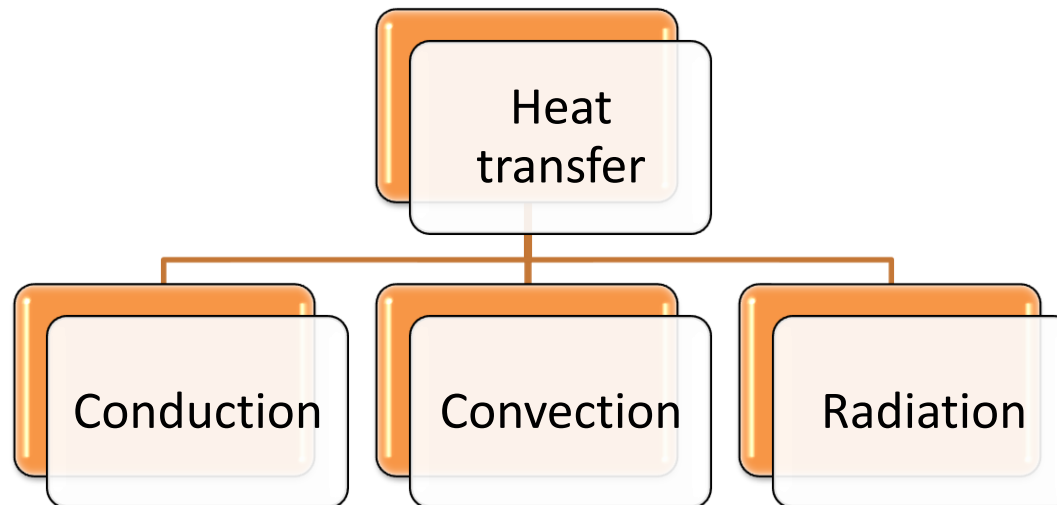




4-5. THERMAL SYSTEMS

Assumptions before the simulation:

1. substances that are characterized by resistance to heat flow have negligible heat capacitance
2. substances that are characterized by heat capacitance have negligible resistance to heat flow





4-5. THERMAL SYSTEMS

For conduction or convection heat transfer $q = K \Delta\theta$

where q = heat flow rate, kcal/sec

$\Delta\theta$ = temperature difference, °C

K = coefficient, kcal/sec °C

The coefficient K is given by

$$K = \frac{kA}{\Delta X}, \quad \text{for conduction}$$

$$= HA, \quad \text{for convection}$$

where k = thermal conductivity, kcal/m sec °C

A = area normal to heat flow, m²

ΔX = thickness of conductor, m

H = convection coefficient, kcal/m² sec °C



4-5. THERMAL SYSTEMS

Thermal Resistance and Thermal Capacitance.

$$R = \frac{\text{change in temperature difference, } ^\circ\text{C}}{\text{change in heat flow rate, kcal/sec}}$$

The thermal resistance for conduction or convection heat transfer is given by

$$R = \frac{d(\Delta\theta)}{dq} = \frac{1}{K}$$

The thermal capacitance C is defined by $C = \frac{\text{change in heat stored, kcal}}{\text{change in temperature, } ^\circ\text{C}} = mc$

where m = mass of substance considered, kg

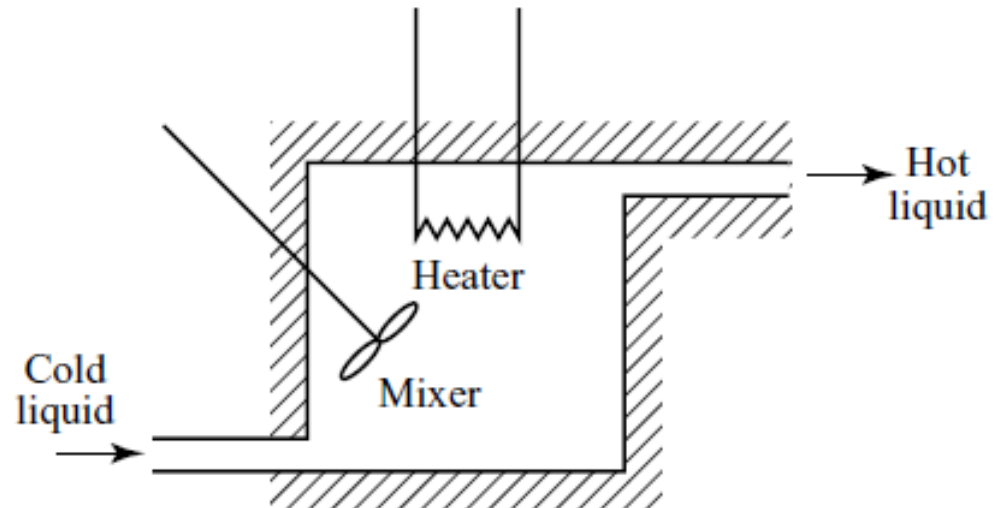
c = specific heat of substance, kcal/kg $^\circ\text{C}$

4-5. THERMAL SYSTEMS

Thermal System.

Assumptions:

1. the tank is insulated to eliminate heat loss to the surrounding air
2. there is no heat storage in the insulation and that the liquid in the tank is perfectly mixed so that it is at a uniform temperature





4-5. THERMAL SYSTEMS

Definitions

$\bar{\theta}_i$ = steady-state temperature of inflowing liquid, °C

$\bar{\theta}_o$ = steady-state temperature of outflowing liquid, °C

G = steady-state liquid flow rate, kg/sec

M = mass of liquid in tank, kg

c = specific heat of liquid, kcal/kg °C

R = thermal resistance, °C sec/kcal

C = thermal capacitance, kcal/°C

\bar{H} = steady-state heat input rate, kcal/sec



4-5. THERMAL SYSTEMS

Case 1: change is only in the heat input

Analysis procedures

Assumptions:

1. the temperature of inlet fluid is constant
2. The heat input rate to the system (heat supplied by the heater) is suddenly changed from \bar{H} to $\bar{H} + h_i$ where h_i represents a small change in the heat input rate
3. the heat outflow rate will then change gradually from \bar{H} to $\bar{H} + h_o$
4. The temperature of the out flowing liquid will also be changed from $\bar{\theta}_o$ to $\bar{\theta}_o + \theta$.

Equations

$$C d\theta = (h_i - h_o) dt$$

$$h_o = Gc\theta$$

$$C = Mc$$

$$C \frac{d\theta}{dt} = h_i - h_o$$

$$R = \frac{\theta}{h_o} = \frac{1}{Gc}$$



4-5. THERMAL SYSTEMS

Combine the previous equations

$$RC \frac{d\theta}{dt} + \theta = Rh_i$$

Take Laplace with I.Cs = 0

$$\frac{\Theta(s)}{H_i(s)} = \frac{R}{RCs + 1}$$



4-5. THERMAL SYSTEMS

Case 2: change is only in the input temperature

$$\bar{\Theta}_i \text{ to } \bar{\Theta}_i + \theta_i \left\{ \begin{array}{l} \bar{H} \text{ to } \bar{H} + h_o \\ \bar{\Theta}_o \text{ to } \bar{\Theta}_o + \theta \end{array} \right.$$

$$C d\theta = (Gc\theta_i - h_o) dt \quad \Rightarrow \quad C \frac{d\theta}{dt} = Gc\theta_i - h_o$$

$$\frac{\Theta(s)}{\Theta_i(s)} = \frac{1}{RCs + 1}$$

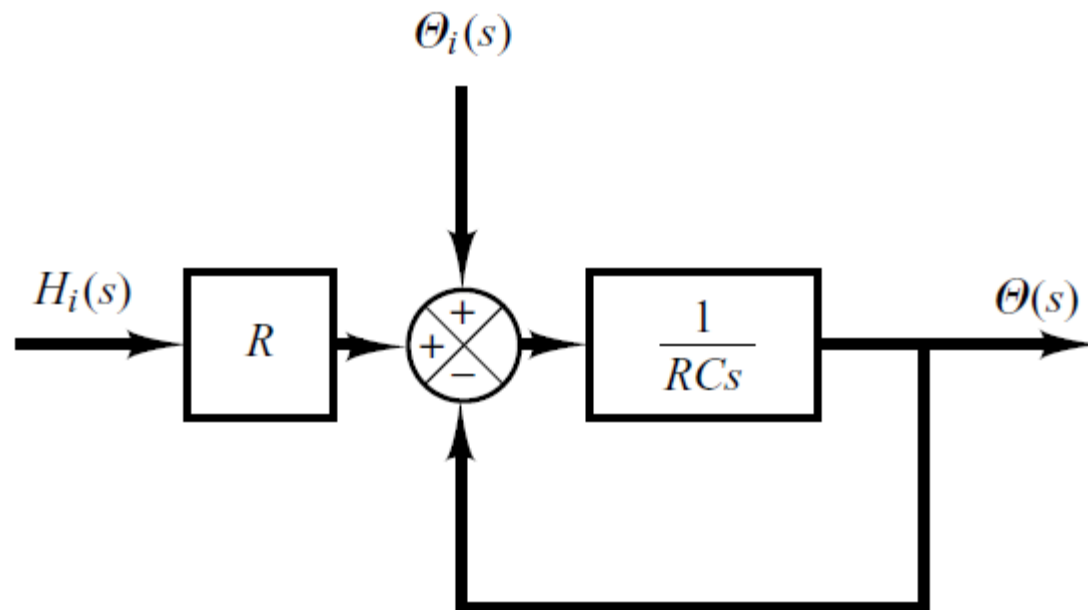
$$RC \frac{d\theta}{dt} + \theta = \theta_i$$



4-5. THERMAL SYSTEMS

Case 3: changes are in both the input temperature and the heat input

$$RC \frac{d\theta}{dt} + \theta = \theta_i + Rh_i$$





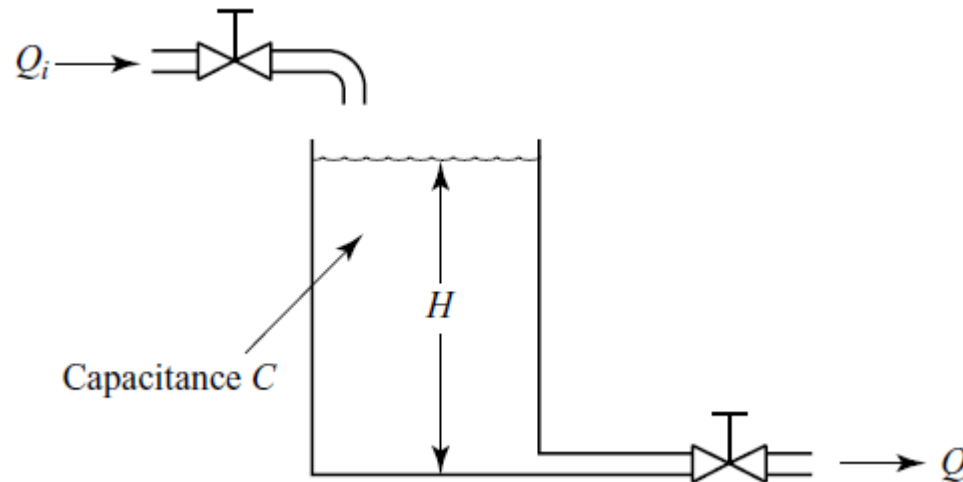
Examples

A-4-1.

In the liquid-level system of Figure 4-27 assume that the outflow rate Q m³/sec through the outflow valve is related to the head H m by

$$Q = K\sqrt{H} = 0.01\sqrt{H}$$

Assume also that when the inflow rate Q_i is 0.015 m³/sec the head stays constant. For $t < 0$ the system is at steady state ($Q_i = 0.015$ m³/sec). At $t = 0$ the inflow valve is closed and so there is no inflow for $t \geq 0$. Find the time necessary to empty the tank to half the original head. The capacitance C of the tank is 2 m².





Examples

Solution. When the head is stationary, the inflow rate equals the outflow rate. Thus head H_0 at $t = 0$ is obtained from

$$0.015 = 0.01\sqrt{H_0}$$

or

$$H_0 = 2.25 \text{ m}$$

The equation for the system for $t > 0$ is

$$-C dH = Q dt$$

or

$$\frac{dH}{dt} = -\frac{Q}{C} = \frac{-0.01\sqrt{H}}{2}$$

Hence

$$\frac{dH}{\sqrt{H}} = -0.005 dt$$



Examples

Assume that, at $t = t_1$, $H = 1.125$ m. Integrating both sides of this last equation, we obtain

$$\int_{2.25}^{1.125} \frac{dH}{\sqrt{H}} = \int_0^{t_1} (-0.005) dt = -0.005t_1$$

It follows that

$$2\sqrt{H} \Big|_{2.25}^{1.125} = 2\sqrt{1.125} - 2\sqrt{2.25} = -0.005t_1$$

or

$$t_1 = 175.7$$

Thus, the head becomes half the original value (2.25 m) in 175.7 sec.

Mathematical Modeling of fluid and thermal systems

Examples

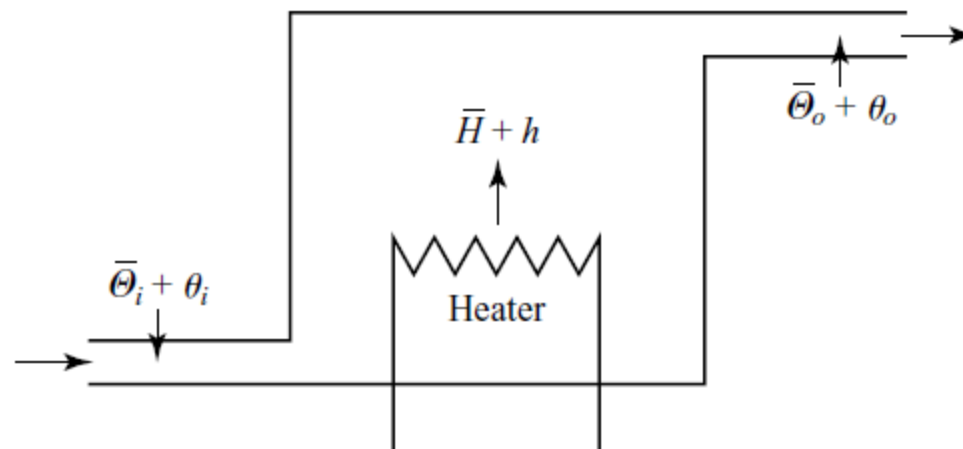




Examples

A-4-10.

Considering small deviations from steady-state operation, draw a block diagram of the air heating system shown in Figure 4-38. Assume that the heat loss to the surroundings and the heat capacitance of the metal parts of the heater are negligible.





Examples

Solution. Let us define

$\bar{\theta}_i$ = steady-state temperature of inlet air, °C

$\bar{\theta}_o$ = steady-state temperature of outlet air, °C

G = mass flow rate of air through the heating chamber, kg/sec

M = mass of air contained in the heating chamber, kg

c = specific heat of air, kcal/kg °C

R = thermal resistance, °C sec/kcal

C = thermal capacitance of air contained in the heating chamber = Mc , kcal/°C

\bar{H} = steady-state heat input, kcal/sec

Let us assume that the heat input is suddenly changed from \bar{H} to $\bar{H} + h$ and the inlet air temperature is suddenly changed from $\bar{\theta}_i$ to $\bar{\theta}_i + \theta_i$. Then the outlet air temperature will be changed from $\bar{\theta}_o$ to $\bar{\theta}_o + \theta_o$.

The equation describing the system behavior is

$$C d\theta_o = [h + Gc(\theta_i - \theta_o)] dt$$

or

$$C \frac{d\theta_o}{dt} = h + Gc(\theta_i - \theta_o)$$



Examples

Noting that

$$G_c = \frac{1}{R}$$

we obtain

$$C \frac{d\theta_o}{dt} = h + \frac{1}{R} (\theta_i - \theta_o)$$

or

$$RC \frac{d\theta_o}{dt} + \theta_o = Rh + \theta_i$$

Taking the Laplace transforms of both sides of this last equation and substituting the initial condition that $\theta_o(0) = 0$, we obtain

$$\Theta_o(s) = \frac{R}{RCs + 1} H(s) + \frac{1}{RCs + 1} \Theta_i(s)$$

